Profile generator for time optimal jerk–limited trajectories able to follow moving targets

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1 Nomenclature and problem setup

We denote by $x(t)$ the motion of the axis, and by $x_c(t)$ the motion of the target. The index c stands for the conveyor moving at constant velocity v_c. The determination of the optimal trajectory is easier to derive in relative motion. The relative position is defined as $x_r(t) = x(t) - x_c(t)$. The symmetrical constraint $|\dot{x}(t)| < v_{\max}$ becomes asymmetrical $-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c$. Relative acceleration $\ddot{x}_r(t)$ and axis acceleration $\ddot{x}(t)$ are the same, since the conveyor is supposed to move at constant velocity. The following section shows how to compute the time–optimal profile for arbitrary initial conditions $x_r(0), \dot{x}_r(0), \ddot{x}_r(0)$ of the relative motion and arbitrary linear target motion $x_c(t) = x_{c0} + v_c\, t$.

2 Determination of jerk switching points in continuous time

Among all functions[1] $x_r(t)$ satisfying

$$x_r(0) = x_{r0},\ \dot{x}_r(0) = v_{r0},\ \ddot{x}_r(0) = a_0$$
$$x_r(T) = 0,\ \dot{x}_r(T) = 0,\ \ddot{x}_r(T) = 0$$

under the constraints

$$-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c,\ |\ddot{x}_r(t)| \leqslant a_{\max}\ \text{ for } t \in [0, T]$$

and

$$|\dddot{x}_r(t)| \leqslant j_{\max}\ \text{ for almost all } t \in [0, T]$$

we are looking for the one for which T is minimal.

The existence of such a function $x_r(t)$ satisfying the initial and final conditions implies

$$v_{r0} + \frac{a_0^2}{2 j_{\max}} \leqslant v_{\max}\ \text{ if }\ a_0 \geqslant 0$$

and

$$v_{r0} - \frac{a_0^2}{2 j_{\max}} \geqslant -v_{\max}\ \text{ if }\ a_0 \leqslant 0.$$

The general profile of the solution has the following structure

$$\begin{aligned} x_r(t) \ &= \ x_{r0} + v_{r0} t + \frac{1}{2} a_0 t^2 + \frac{1}{6}\, c\, t^3 - \frac{1}{6}\epsilon(t - t_1)\, c\, (t - t_1)^3 - \frac{1}{6}\epsilon(t - t_2)\, c\, (t - t_2)^3 \\ &+ \ \frac{1}{6}\epsilon(t - t_3)\, c\, (t - t_3)^3 - \frac{1}{6}\epsilon(t - t_4)\, c\, (t - t_4)^3 + \frac{1}{6}\epsilon(t - t_5)\, c\, (t - t_5)^3 + \frac{1}{6}\epsilon(t - t_6)\, c\, (t - t_6)^3 \end{aligned}$$

[1] two times continuously derivable on the interval $[0, T]$ and whose third derivative is essentially bounded in the sense of distributions

where $c = \pm j_{\max}$, $0 \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant t_4 \leqslant t_5 \leqslant t_6 \leqslant T$ and $\epsilon(\cdot)$ denotes the Heaviside step function.

case 1:
If $a_0 \geqslant 0$ and if $x_{r0} = \dfrac{a_0^3}{6j_{\max}^2}$ and $v_{r0} = -\dfrac{a_0^2}{2j_{\max}}$ then $x(t) = \dfrac{(a_0 - j_{\max}t)^3}{6j_{\max}^2}$ and $T = \dfrac{a_0}{j_{\max}}$.
If $a_0 < 0$ and if $x_{r0} = \dfrac{a_0^3}{6j_{\max}^2}$ and $v_{r0} = \dfrac{a_0^2}{2j_{\max}}$ then $x(t) = \dfrac{(a_0 + j_{\max}t)^3}{6j_{\max}^2}$ and $T = -\dfrac{a_0}{j_{\max}}$.

case 2:
Otherwise we pose $t_2 = t_1$, $t_4 = t_3$, $t_6 = t_5$ and we determine $c \in \{-j_{\max}, j_{\max}\}$ such that the system of equations

$$\begin{cases} x_r(T) &= 0 \\ \dot{x}_r(T) &= 0 \\ \ddot{x}_r(T) &= 0 \end{cases}$$

with unknowns t_1, t_5 and T has a solution satisfying $0 \leqslant t_1 \leqslant t_5 \leqslant T$.
If $|\ddot{x}_r(t_1)| \leqslant a_{\max}$ and $|\ddot{x}_r(t_5)| \leqslant a_{\max}$ and if $-v_{\max} - v_c \leqslant \dot{x}_c(t) \leqslant v_{\max} - v_c$ for $t \in [0, T]$ we constructed the wanted profile. In case 2, the acceleration constraints are not active.

case 3:
Otherwise, if $\ddot{x}_r(t_5) > a_{\max}$ we necessarily have $c = -j_{\max}$. We pose $t_2 = t_1$, $t_4 = t_3$ and we compute the solution of the following system of equations

$$\begin{cases} \ddot{x}_r(t_5) &= a_{\max} \\ x_r(T) &= 0 \\ \dot{x}_r(T) &= 0 \\ \ddot{x}_r(T) &= 0 \end{cases}$$

with unknowns t_1, t_5, t_6 and T satisfying $0 \leqslant t_1 \leqslant t_5 \leqslant t_6 \leqslant T$.
If $\ddot{x}_r(t_1) \geqslant -a_{\max}$ and if $-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c$ for $t \in [0, T]$ we constructed the wanted profile.

case 4:
Otherwise, if $\ddot{x}_r(t_1) < -a_{\max}$ we pose $t_4 = t_3$ and we compute the solution of the following system of equations

$$\begin{cases} \ddot{x}_r(t_1) &= -a_{\max} \\ \ddot{x}_r(t_5) &= a_{\max} \\ x_r(T)_r &= 0 \\ \dot{x}_r(T) &= 0 \\ \ddot{x}_r(T) &= 0 \end{cases}$$

with unknowns t_1, t_2, t_5, t_6 and T satisfying $0 \leqslant t_1 \leqslant t_2 \leqslant t_5 \leqslant t_6 \leqslant T$.
If $-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c$ for $t \in [0, T]$ we constructed the wanted profile.

case 5:
Otherwise, if $\ddot{x}_r(t_5) < -a_{\max}$ we necessarily have $c = j_{\max}$. We pose $t_2 = t_1$, $t_4 = t_3$ and we compute the solution of the following system of equations

$$\begin{cases} \ddot{x}_r(t_5) &= -a_{\max} \\ x_r(T) &= 0 \\ \dot{x}_r(T) &= 0 \\ \ddot{x}_r(T) &= 0 \end{cases}$$

with unknowns t_1, t_5, t_6 and T satisfying $0 \leqslant t_1 \leqslant t_5 \leqslant t_6 \leqslant T$.
If $\ddot{x}_r(t_1) \leqslant a_{\max}$ and if $-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c$ for $t \in [0, T]$ we constructed the wanted profile.

case 6:

Otherwise, if $\ddot{x}_r(t_1) > a_{\max}$ we pose $t_4 = t_3$ and we compute for $c = j_{\max}$ the solution of the following system of equations

$$\begin{cases} \ddot{x}_r(t_1) &= a_{\max} \\ \ddot{x}_r(t_5) &= -a_{\max} \\ x_r(T) &= 0 \\ \dot{x}_r(T) &= 0 \\ \ddot{x}_r(T) &= 0 \end{cases}$$

with unknowns t_1, t_2, t_5, t_6 and T satisfying $0 \leqslant t_1 \leqslant t_2 \leqslant t_5 \leqslant t_6 \leqslant T$.
If $-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c$ for $t \in [0, T]$ we constructed the wanted profile. Case 6 is identical to case 4 except for a change of sign in c and $a_{\max}$.

case 7:
Otherwise, if $|\ddot{x}_r(t_5)| \leqslant a_{\max}$ and $\ddot{x}_r(t_1) < -a_{\max}$ we necessarily have $c = -j_{\max}$. We pose $t_6 = t_5$

$$\begin{cases} \ddot{x}_r(t_1) &= -a_{\max} \\ x_r(T) &= 0 \\ \dot{x}_r(T) &= 0 \\ \ddot{x}_r(T) &= 0 \end{cases}$$

with unknowns t_1, t_2, t_5 and T satisfying $0 \leqslant t_1 \leqslant t_2 \leqslant t_5 \leqslant T$.
If $-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c$ for $t \in [0, T]$ we constructed the wanted profile.

case 8:
Otherwise, if $|\ddot{x}_r(t_5)| \leqslant a_{\max}$ and $\ddot{x}_r(t_1) > a_{\max}$ we necessarily have $c = j_{\max}$. We pose $t_6 = t_5$ and we compute the solution of the following system of equations

$$\begin{cases} \ddot{x}_r(t_1) &= a_{\max} \\ x_r(T) &= 0 \\ \dot{x}_r(T) &= 0 \\ \ddot{x}_r(T) &= 0 \end{cases}$$

with unknowns t_1, t_2, t_5 and T satisfying $0 \leqslant t_1 \leqslant t_2 \leqslant t_5 \leqslant T$.
If $-v_{\max} - v_c \leqslant \dot{x}_r(t) \leqslant v_{\max} - v_c$ for $t \in [0, T]$ we constructed the wanted profile. Case 8 is identical to case 7 except for a change of sign in c and $a_{\max}$.

cases where $v_{\max}$ is active:
Otherwise, if there exist $\bar{t} \in [0, T]$ such that $\dot{x}_r(\bar{t}) > v_{\max} - v_c$ or such that $\dot{x}_r(\bar{t}) < -v_{\max} - v_c$ we restart the algorithm by completing the system of equations with the following equations

$$\begin{cases} \ddot{x}_r(t_3) &= 0 \\ \dot{x}_r(t_3) &= v_{\max} - v_c \end{cases}$$

or

$$\begin{cases} \ddot{x}_r(t_3) &= 0 \\ \dot{x}_r(t_3) &= -v_{\max} - v_c \end{cases}$$

Potentially admissible solutions for case 2 are given by

$$t_1 = \frac{2(a_0^2 - 2cv_{r0}) - 4a_0c\tau + c^2\tau^2}{4c^2\tau},\ t_5 = t_1 + \frac{\tau}{2},\ T = -\frac{a_0}{c} + \tau$$

where $\tau > 0$ satisfies the relation

$$3c^4\tau^4 - 24c^2(a_0^2 - 2cv_{r0})\tau^2 + 32c(a_0^3 - 3a_0cv_{r0} + 3c^2x_{r0})\tau - 12(a_0^2 - 2cv_{r0})^2 = 0.$$

If the velocity constraint is active, the computation of the solution only needs solving polynomials of degree at most 2.